Conserved lattice gas model with infinitely many absorbing states in one dimension

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The conserved lattice gas model with infinitely many absorbing states is studied on a chain and on a ladder. In both one-dimensional lattices it exhibits a phase transition from an absorbing phase to an active state. The model defined on a chain is solved exactly and shows a critical behavior with classical critical exponents. However, the model defined on a ladder shows a critical behavior, obtained from numerical simulation, that places the model in the same universality class as the Manna model.

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I. INTRODUCTION

Lattice models have been widely used to describe self-organized criticality (SOC). Among them the sandpile models, which include the models introduced by Bak, Tand, and Wiesenfeld [1], by Dhar [2], and by Manna [3], have attracted attention for being models which exhibit SOC properties. Other similar models include the activated random walkers model [4,5], sandpile models with height restrictions [6], and the threshold transfer process [7,8]. Another class of models exhibiting SOC properties is the class of the exclusion models in which at most one particle may occupy a site of the lattice. In this class we find the conserved lattice gas (CLG) models [7,9].

The basic ingredient that relates these models to SOC is the presence of infinitely many absorbing states [10,11]. Other lattice gas models show infinitely many absorbing states such as the pair contact process [12]. The main property that distinguishes the CLG models is the conservation of particles and for this reason they are also known as fixed-energy sandpile models. The CLG model is defined on a lattice where active particles may jump to neighboring empty sites. A particle is active if it has at least one of its nearest neighbor sites occupied. Only active particles are allowed to move. Isolated particles do not move and a state made up by isolated particles is therefore an absorbing state. If the number of particles is low enough then the model exhibits many absorbing states. As we vary the number of particles the CLG model exhibits a continuous phase transition from an absorbing state to an active state where the number of active particles is nonzero.

The connection between SOC and models with infinitely many absorbing states such as the CLG model can be understood as follows. The process is started in a finite lattice with a certain number of particles. If there are no active particles or if the system falls into an absorbing configuration then a particle is added to the system. If there are active particles the process is run until a particle falls off the lattice. The mechanism of adding and removing particles is repeated several times. This procedure, which is actually used in SOC models, amounts to removing a particle if the system is in the active state and introducing a particle if the system is in the absorbing state, driving thus the system toward the critical state. An avalanche corresponds to the moves of the active particles. The time it takes for the system to fall into an absorbing state, when a particle is added, is taken as a measure of the lifetime of an avalanche.

Studies of the continuous time version of the CLG model in hypercubic lattice of dimensions $d=2$, 3, 4, and 5 [9] have given evidence that it belongs to the same universality class as the Manna model. However, in a one-dimensional chain the CLG model does not show the critical behavior we would expect for a model in the Manna universality class. Actually, on a chain the CLG model exhibits a classical critical behavior as we show here by exactly solving the model [13]. By classical behavior we mean the same behavior obtained when a mean-field treatment is used. The exact solution on a chain is carried out here by showing that all configurations of the active state are equally probable constituting in this sense a microcanonical ensemble. By changing to a grand canonical ensemble the model can then be exactly solved by means of a transfer matrix technique. The exponent related to the order parameter was found to be the classical one, namely $\beta=1$.

Nonclassical critical behavior can, however, be found in one dimension by defining the CLG model in another lattice. On a ladder, for instance, we have found, by numerical simulations, a nonclassical critical exponent. The exponent related to the order parameter was found to be $\beta=0.40(1)$ consistent with a model belonging to the Manna universality class [5,6,8].

II. CLG MODEL ON A CHAIN

Let us consider a one-dimensional chain with L sites and periodic boundary conditions. Each site can be empty or occupied by just one particle. The total number of particles is N. The particles are classified according to the occupancy of its two nearest neighbor sites. If both sites are empty the particle is said to be isolated; if both sites are occupied the particle is blocked; and if one site is empty and the other occupied the particle is active. The dynamical rules are as follows. At each time step a particle is chosen at random. If it is an active particle it is moved to the empty nearest neighbor site. Isolated and blocked particles do not move. The possible transitions are as follows:

$$110 \to 101, \quad 011 \to 101, \tag{1}$$

and they have the same rates. Empty and occupied sites are represented by 0 and 1, respectively.

Let us examine the evolution of the number of nearest neighbor pairs of occupied sites N_{11} and the number of nearest neighbor pairs of empty sites N_{00}. According to the rules these numbers never increase. To prove this property it is enough to look at the possible transitions related to four sites in a row. They are

$$1100 \to 1010, \quad 0011 \to 0101, \tag{2}$$

$$1101 \to 1011, \quad 1011 \to 1101, \tag{3}$$

and they have all the same rates. The transitions in Eq. (3) conserve the number of 11 pairs whereas the transitions in Eq. (2) decreases the number of 11 pairs as well as the number of 00 pairs.

To proceed in our analysis, we have to distinguish whether the number of particles N is smaller or larger than half the number of sites L. In the first case $N<L/2$, it is always possible to set up absorbing configurations by spreading out the particles over the lattice so that any particle be isolated. Absorbing states are stable against a perturbation in which a particle is moved next to another one. Indeed, this perturbation will create a 11 pair and a 00 pair next to each other, but the rules (2) will destroy both pairs reestablishing the absorbing state. We may conclude that, for $N<L/2$, the quantity N_{11} decreases and vanishes in the stationary absorbing state, whereas N_{00} decreases but remains finite in the stationary absorbing state.

In the second case, namely $N>L/2$, there is no absorbing configurations. Any stationary state is active because the system has always at least one pair of nearest neighbor occupied sites, and therefore at least two active particles. In this case, N_{11} decreases but must remain finite in the stationary state. On the other hand, N_{00} decreases and we assume it vanishes in the stationary state. This amounts to saying that the stationary state is devoid of pairs of nearest neighbor empty sites. We call these states isolated-vacancy (IV) states. If such a state is perturbed by moving a particle in a way that a pair of nearest neighbor empty sites be created then according to rules in Eq. (2) the IV state will be restored.

III. ACTIVE STATE

Within the subspace of IV states, the pertinent rules are those corresponding to the two transitions given in Eq. (3). These two transitions transform a given IV state into another IV state. Since they are reverse of each other and have the same rate, the process is microscopically reversible (obeys detailed balance) within the subspace of IV states. It follows immediately that the stationary probability is the same for all IV configurations with the same number of particles. Since any IV configuration can be reached from any other by the rules, then all IV configurations with a given number of particles make up the stationary state, with the same probability. In other words the IV configurations make up a Gibbs microcanonical ensemble with a fixed number of particles.

In the regime $N>L/2$, the configurations that are not IV are transient and the stationary active state is constituted by IV states only. An important property of the IV states is that all IV states with N particles have the same number of nearest neighbor pair of occupied sites. In other words, N_{11} is a conserved quantity. Indeed, for the IV states only the transitions in Eq. (3) are effective and these two transitions conserve the number of 11 pairs. To calculate N_{11} we can use therefore any configuration. The simplest configuration is the one in which the particles are either isolated or belong to a single cluster with more than one particle. This cluster has $(N_{11}+1)$ particles so that the number of isolated particles must be $(L-N_{11}-2)/2$. The sum of these number equals the number of particles N so that $N_{11}=(2N-L)$. The density of 11 pairs $\rho_{11}=N_{11}/L$ is related to the density of particles $\rho=N/L$ by

$$\rho_{11} = 2\rho - 1. \tag{4}$$

The quantity ρ_{11} may be understood as an order parameter in which case it follows that the critical density is $\rho_c=1/2$ and the critical exponent $\beta=1$.

Let us consider a given IV configuration and denote by N_C the number of cluster of particles of more than one particle. Only the first and the last particle of a cluster are active particles. The other particles of the cluster are blocked particles. Therefore the number of active particles $N_A=2N_C$ is twice the number of clusters. It is easy to see that the number of pairs of nearest neighbor occupied sites N_{11} is related to N_C and to the total number of blocked particles N_B by $N_{11}=N_B+N_C$. Therefore $N_B+N_A/2$ is a conserved quantity within the subspace of IV states. Although N_{11} is a conserved quantity within the subspace of the IV states, the number N_A of active particles is not. To determine N_A as a function of the number of particles N we will examine the properties of the IV states.

IV. GRAND-CANONICAL ENSEMBLE

To determine the properties of active state of the CLG model in a chain it is convenient to change from the ensemble with a fixed number of particle to a grand canonical ensemble with fluctuating number of particles. To this end we introduce the grand canonical partition function

$$Z = \sum_n W_n z^n, \tag{5}$$

where z is the activity and W_n denotes the number of IV configurations with n particles. When $n<L/2$, $W_n=0$. The probability of a IV configuration $\mathcal{C}$ in the grand canonical ensemble is

$$P(\mathcal{C}) = \frac{1}{Z} z^n, \tag{6}$$

where n is the number of particles in $\mathcal{C}$. The partition function Z can be calculated by

$$Z = \sum_{\mathcal{C}} z^n, \tag{7}$$

where the summation is over all IV configuration $\mathcal{C}$.

To calculated Z we use the transfer matrix approach according to which Z is given by

$$Z = \mathrm{Tr}\, T^L. \tag{8}$$

The elements of the transfer matrix T connects two consecutive sites of the lattice. Since IV configurations have no double vacancy it follows that $T_{00}=0$. The other elements are $T_{11}=z$, $T_{10}=\sqrt{z}$, and $T_{01}=\sqrt{z}$. The eigenvalues of the T are

$$\lambda_\pm = \frac{z \pm \sqrt{z^2+4z}}{2}, \tag{9}$$

so that

$$Z = \lambda_+^L + \lambda_-^L. \tag{10}$$

The density of particles ρ is obtained by $\rho = (1/L) z\, \partial \ln Z/\partial z$ which, in the thermodynamic limit, gives the following relation between ρ and z:

$$\rho = \frac{\lambda_+ + 1}{\lambda_+ + 2} = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{z}{z+4}}, \tag{11}$$

so that the critical density $\rho_c = 1/2$ is reached when $z \to 0$. Let us determine the density ρ_{11} of the pairs 11 which is also the probability P_{11}. Since $P_{11}+P_{10}=P_1$ and $P_{10}+P_{00}=P_0$, and taking into account that $P_{00}=0$ it follows that $P_{11}=2P_1-1$, that is, $\rho_{11}=2\rho-1$ which is the result already obtained.

To determine the density ρ_a of active states we calculate the probability P_{110} and use the relation $\rho_a = P_{110}+P_{011} = 2P_{110}$. This probability is given by

$$P_{110} = \frac{1}{Z}\,\mathrm{Tr}\, QRT^{L-2}, \tag{12}$$

where the matrices Q and R are such that their only nonzero entries are $Q_{11}=T_{11}$ and $R_{10}=T_{10}$. Computing the trace we get, in the thermodynamic limit, the following relation between ρ_a and z:

$$\rho_a = \frac{2\lambda_+}{(\lambda_+ + 1)(\lambda_+ + 2)}. \tag{13}$$

The relation between the density of active particles and the density of particles is then

$$\rho_a = \frac{2}{\rho}(2\rho - 1)(1-\rho), \tag{14}$$

and is shown in Fig. 1. Therefore the order parameter ρ_a vanishes at the critical density $\rho_c = 1/2$ with an exponent $\beta = 1$. It is worth mentioning that the density of blocked particles $\rho_b = \rho_{11} - \rho_a/2$ is given by

$$\rho_b = \frac{1}{\rho}(2\rho - 1)^2, \tag{15}$$

so that it becomes negligible as we approach the critical density $\rho_c = 1/2$.

The correlation length ξ can also be determined from the ratio between the two eigenvalues of T. It is given by $\xi^{-1} = |\ln|\lambda_-|/\lambda_+|$. As one approaches the critical point, $z \to 0$, we get $\xi = z^{-1/2}$ so that

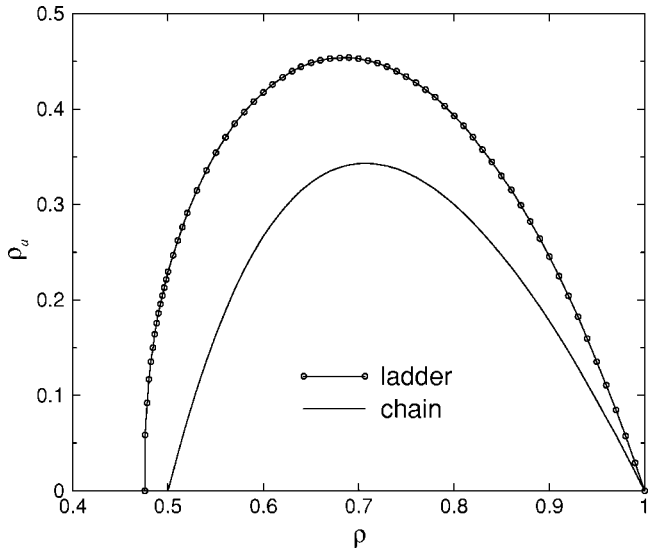

FIG. 1. Density ρ_a of active particles as a function of the density ρ of particles for the CLG model defined on a chain and on a ladder. It is shown the exact result (14) for the chain and numerical simulations on a lattice of size $L=2000$ for the ladder.

$$\xi = \frac{1}{2}\left(\rho - \frac{1}{2}\right)^{-1/2}, \tag{16}$$

given the exponent $\nu = 1/2$.

V. CRITICAL STATE AND AVALANCHES

The avalanches that are connected to SOC correspond to moves of active particles in the critical state. In the supercritical regime the moves never stop if the system is infinite. If the system is finite particles will eventually fall off, driving the system to the critical state by decreasing the number of particles. In the subcritical regime the lifetime will be finite. In this case a particle is added to the system driving it to the critical state by increasing the number of particles. The relevant moves, an avalanche, are then those moves occurring at the critical state. The probability density of the lifetimes, at the critical point, behaves as

$$P(t) \sim t^{-\alpha} \tag{17}$$

for large values of t.

To determine the probability $P(t)$ for the CLG model in a chain we proceed as follows. The critical state occurs when $N = L/2$, where we are assuming that the number of sites L is even. In this case the stationary state is an absorbing configuration in which the particles occupy every other site of the chain so that all particles and all vacancies are isolated. The system is perturbed by moving a particle to one of its nearest neighbor empty site so that it will be close to another particle. These two particles become active and the system evolves in time until the originally perturbed particle comes back to its original site. The perturbed critical state consists of just one pair of active particles that follows a random walk. In one dimension the time it takes for a walker to return to its original place is distributed according to

$$P(t) \sim t^{-1/2} \tag{18}$$

which is then identified with the distribution of the lifetime of the avalanches. Therefore $\alpha = 1/2$.

VI. CLG MODEL ON A LADDER

The CLG model defined on a chain is such that an active particle has always just one neighboring empty site. It is possible to relax this constraint while still remaining in one dimension by studying the CLG model on a ladder. Each site of the ladder has three nearest neighbor sites so that an active particle may have one or two neighboring empty sites. We have performed numerical simulations with dynamical rules defined as follows. At each time step a particle is selected at random. If it is active, then it is moved, with equal probability, to one of the three neighboring sites. If the chosen neighboring site is already occupied the move is not carried out and the particle remains in its place.

In Fig. 1 we show a plot of the density of active particles as a function of the density of particles for a ladder with size $L = 2000$. The critical density was obtained by assuming the behavior $\rho_a \sim (\rho - \rho_c)^{\beta}$. A log-log plot of ρ_a vs ρ for several trial values of ρ_c will determined the critical value. The best fitting to a straight line gives $\rho_c = 0.4755(2)$ and $\beta = 0.40(1)$ as shown in Fig. 2. This numerical value of β is consistent with corresponding values of other one-dimensional models in the universality class of the Manna model [5,6,8].

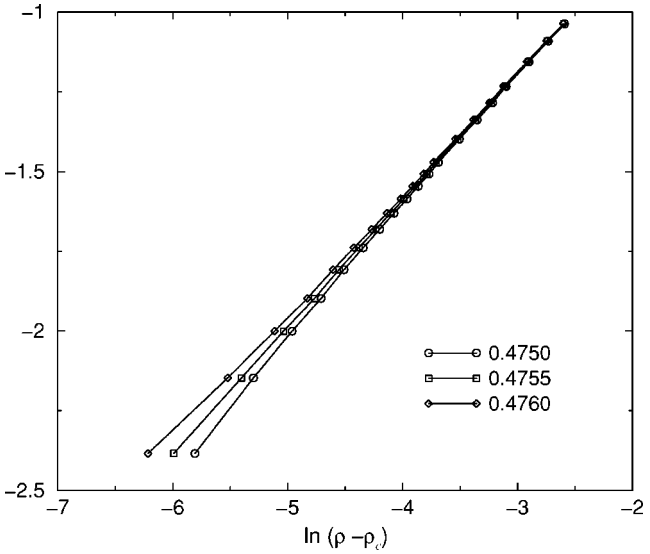

FIG. 2. Log-log plot of the density ρ_a of active particles as a function of the deviation of the density ρ of particles from its critical value ρ_c for the CLG model defined on a ladder. Several trial values of ρ_c are shown. The best fitting to a straight line gives $\rho_c = 0.4755$ and $\beta = 0.40$.

VII. CONCLUSION

We have studied one-dimensional CLG models defined on a chain and on a ladder. We have shown that the stationary active state of the chain model is made up of configurations devoid of pairs of nearest neighbor empty sites. The empty sites shows up only as isolated vacancies. This property is sufficient to show that all configurations of this type are equally probable defining thus a Gibbs microcanonical ensemble. The change to a grand-canonical ensemble allowed then an exact solution which reveals a critical behavior with classical exponents. The numerical investigation of the model CGL defined on a ladder, on the other hand, reveals a critical behavior that puts this model in the same universality class as the Manna model.

It is worth mentioning that it is possible to define d-dimensional models with properties similar to the CLG chain model studied here. This may be done, for instance, in a d-dimensional hypercubic lattice, by defining an active particle as the one which has all but one nearest neighbor sites occupied. An active particle has just one way out of its place. The active state will then be composed by isolated vacancy configurations. The quantities N_{00} and N_{11} will decreases monotonically, the first vanishing in the active state, the second in the absorbing state. In the stationary state, these restricted models defined on a d-dimensional hypercubic lattice are equivalent to an equilibrium system of hard-core particles in which a particle excludes the presence of particles on its nearest neighbor sites. The equivalence is established by mapping the empty site of the restricted CLG model into a hard-core particle.

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